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G-SPECTRAL ESTIMATION.(U)
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(6) G-SPECTRAL ESTIMATION

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ABSTRACT. In [1], 1971, a transformation referred to as the G-Transformation was introduced by H. L. Gray, T. A. Atchison and G. V. McWilliams. The G-transformation ~~was~~ has been shown to be of value in evaluating improper integrals and was especially effective in those cases where the integrand behaved like a linear combination of exponentials with real or complex arguments. These ideas have more recently ~~([2,3] 1976-77)~~ been utilized to define a spectral estimator referred to as the G-Spectral Estimator. In this paper the nature of the G-transform and G-Spectral estimator are discussed and it is pointed out that such an estimator is most appropriate for processes whose autocorrelations behave as a linear combination of exponentials with real or complex arguments.

Let us begin with a few necessary definitions.

Definition 1. Let a be constant and let

$$F(t) = \int_a^t f(x)dx,$$

where $F(\infty)$ is assumed finite.

Then we define the G-transform by

$$G[F(t);h,n] = \frac{\begin{vmatrix} F(t) & F(t+h) & \cdots & F(t+nh) \\ f(t) & f(t+h) & \cdots & f(t+nh) \\ \vdots & & & \\ f(t+(n-1)h) & & \cdots & f(t+(2n-1)h) \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ f(t) & f(t+h) & \cdots & f(t+nh) \\ \vdots & & & \\ f(t+(n-1)h) & & \cdots & f(t+(2n-1)h) \end{vmatrix}} \tag{1}$$

Properly defining G in the indeterminate case, see [1], it can be shown that if f satisfies the differential equation

$$f^{(m)}(x) + a_1 f^{(m-1)}(x) + \cdots + a_m f(x) = 0 \tag{2}$$

$x \in (t,\infty)$, then

$$G[F(t);h,n] \equiv F(\infty) \tag{3}$$

for all $h > 0$, $t \geq a$ and $n \geq m$.

For brevity we will limit ourselves to integrals whose limits become infinite in only the positive direction. Since the G-transformation is nonlinear it is not clear from (1) how it should be defined for integrals whose limits become infinite in both directions. Thus we include the following definition.

Definition 2. We define

$$G\left[\int_{-t}^{t} f(x)dx;h,n\right] = G\left[\int_{0}^{t}[f(x)+f(-x)]dx;h,n\right] \tag{4}$$

From Definition 2 and our previous remarks it follows that if $f(x)+f(-x)$ satisfies a differential such as (2) for some n, then

$$G\left[\int_{-t}^{t} f(x)dx;h,n\right] \equiv \int_{-\infty}^{\infty} f(x)dx,$$

for all $t \geq 0$ and $h > 0$.

Since our interest here is spectral estimation and since we will restrict ourselves to real processes we will henceforth only consider symmetric integrands. All of our results will however, with obvious modifications, follow from Definition 2 when this is not the case, see [3]. That is the G-transform method for estimating (or approximating) Fourier transforms is equally applicable to nonsymmetric integrands. We are now in a position to define the G-Spectral Estimator.

Definition 3. Let $\hat{R}(\tau;T)$ be an estimate, based on a record of length T, for the autocorrelation, $R(\tau)$, of real stochastic process with spectral density $S(w)$. Then we define the G-Spectral estimator $\hat{S}(w;T)$ by

$$\hat{S}(w;T) = G\left[2\int_0^{T_0} \hat{R}(\tau;T)\cos 2\pi w\tau d\tau;h,n\right] \tag{5}$$

In (5) the quantities n, T_0, h are constants which must be determined in order to calculate $\hat{S}(w;T)$. A discussion of that problem is beyond the scope of this paper. A complete solution to the problem is however given in [3], where a computer program for calculating $\hat{S}(w;T)$ is given. In most instances $1 < T_0 << T$, $h = 1$, and n is not actually needed. Again see [3]. The spectral estimator defined by (5) enjoys many properties which most other methods do not. We list a few, which subject to mild conditions can be shown when n is properly chosen, and $R(\tau)$ satisfies an equation such as (2).

i) $\lim_{T\to\infty} \hat{S}(w;T) = S(w)$,*

ii) if $\hat{R}(\tau;T) = R(\tau)$ when $0 \leq \tau \leq T_0 < T$, $S(w) \equiv \hat{S}(w;T)$

iii) for most cases $T_0 + (2n-1)h << T$.

* Here the limit can be interpreted in whatever sense is appropriate.

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In essence property (i) states that when $\hat{R}(\tau;T)$ is consistent for $R(\tau)$, $\hat{S}(w;T)$ is consistent for $S(w)$. Property (ii) states that if $R(\tau)$ satisfies an equation such as (2) (the coefficients of that equation are of course not known nor are they assumed known) then there is no error in the approximation due to leakage, i.e. the finite record. Finally property (iii) is listed simply to point out that the sample autocorrelation will normally need be calculated at considerable fewer lags in the calculation of $\hat{S}(w;T)$ than other methods which make use of $\hat{R}(\tau;T)$.

We now give three examples and some comparisons. In the final two examples we use discrete data and the trapezoid approximation to the integrals in (1). When this is done the proper G-transform will yield the discrete Fourier transform of the corresponding discrete autocorrelation, i.e. the leakage error will still be eliminated. A more complete discussion will be given in [3].

Example 1. Consider the continuous autoregressive process defined by

$$\frac{dx}{dt} + \alpha x = z(t)\ ,\ -\infty < t < \infty,$$

where $z(t)$ is white noise.

In this case $R(\tau) = e^{-\alpha|\tau|}$ and for $t \geq 0$ and $h > 0$

$$G_2\left[2\int_0^t e^{-\alpha x}\cos 2\pi wx\,dx; h\right] \equiv \frac{2\alpha}{\alpha^2+(2\pi w)^2} = S(w).$$

Example 1 is of course trivial and only meant to demonstrate the exactness property of the G-transform. A result which is less trivial and suggested by this example and our comments above is the following.

Example 2. Let $z(t)$, $t \in [0\bar{+}1,\ldots)$ be white noise and let $\{x(t)\}$ be the discrete autoregressive process

$$x(t) - \beta x(t-1) = z(t).$$

Then

$$R(k) = \beta^{|k|}$$

and

$$S(w) = \frac{1-\beta^2}{1+\beta^2-2\beta\cos 2\pi w}\ ,\ |w| \leq \frac{1}{2}.$$

Thirty realizations of 100 observations were generated from this process and $S(w)$ was estimated by the Bartlett, Tukey and Parzen Window methods*, Burg's MEM, Parzen's Autoregressive Spectral estimator (ARSPEC) and the G-Spectral estimator. The summary statistics are given below. The numbers in parentheses show the range of the number of lags for which the sample autocorrelation was needed except in Burg method where it is the range of the lengths of the prediction error filter.

SUMMARY STATISTICS

(w)	S(w)	Estimator	Sample Mean	Sample MSE
0.00	19.000	*BARTLETT(15)	7.731	129.560
		TUKEY(13)	7.724	128.775
		PARZEN(19)	7.964	123.953
		ARSPEC(1-8)	14.514	85.275
		MEM(1-10)	15.152	76.627
		G(2-6)	15.886	57.839
0.06	1.393	BARTLETT	2.684	2.078
		TUKEY	3.173	3.328
		PARZEN	2.999	2.838
		ARSPEC	1.629	.309
		MEM	1.764	.856
		G	2.146	2.031
0.24	.112	BARTLETT	.230	.0230
		TUKEY	.162	.0109
		PARZEN	.127	.0109
		ARSPEC	.131	.0019
		MEM	.130	.0041
		G	.143	.0069
0.50	.053	BARTLETT	.104	.0042
		TUKEY	.070	.0019
		PARZEN	.099	.0036
		ARSPEC	.064	.0006
		MEM	.062	.0012
		G	.070	.0014

Example 3. Consider the ARMA (3,2) process

$$X_t = 1.5X_{t-1} - 1.21X_{t-2} + .455X_{t-3} + Z_t$$
$$- .175Z_{t-1} + .8Z_{t-2}.$$

Thirty realizations of length 400 points were generated from the above process and the various spectral estimators were again calculated. The summary statistics are as follows.

SUMMARY STATISTICS

(w)	S(w)	Estimator	Sample Mean	Sample MSE
0.00	.017	BARTLETT(38)	.065	.00238
		TUKEY(33)	.023	.00014
		PARZEN(46)	.018	.00028
		ARSPEC(7-18)	.023	.00015
		MEM(8-20)	.014	.00004
		G(4-14)	.018	.00019
0.16	4.063	BARTLETT	3.438	.6718
		TUKEY	3.615	.4314
		PARZEN	3.598	.4619
		ARSPEC	3.907	1.1851
		MEM	3.771	1.2200
		G	4.022	.1924
0.18	4.480	BARTLETT	4.079	.8192
		TUKEY	4.226	.6353
		PARZEN	4.224	.6710
		ARSPEC	5.142	3.7695
		MEM	4.624	2.5005
		G	4.815	.7261
0.50	.319	BARTLETT	.366	.0172
		TUKEY	.349	.0154
		PARZEN	.347	.0158
		ARSPEC	.376	.0231
		MEM	.361	.0252
		G	.336	.0083

* The window methods here and below are better than would be expected in practice since several different length windows were used and the best result is given here.

The above example demonstrates that the G-Spectral estimator tends to have a smaller mean square error (MSE) than the other methods (this is not clear in the table for the window methods since the best window was used which of course could not be done in practice). In some cases it is significantly smaller, in others it is not. More extensive examples however do demonstrate that in many cases the G-spectral estimate is significantly better than the others considered here and hence should be a useful addition to existing methods. For further examples the reader is referred to [3].

As a final comment it should be remarked that the application of the G-spectral estimator is not limited to processes whose autocorrelation satisfy an equation such as (2). Such processes were only considered here since theoretical results suggest that this is the setting in which the method should be most effective. A more extensive investigation would of course be interesting and hopefully forthcoming in the future.

References

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